

Electromagnetics close beyond the critical state: thermodynamic prospect

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Abstract. A theory for the electromagnetic response of type-II superconductors close beyond the critical state is presented. Our formulation relies on general physical principles applied to the superconductor as a thermodynamic system. Equilibrium critical states, externally driven steady solutions, and transient relaxation are altogether described in terms of free energy and entropy production. This approach allows a consistent macroscopic statement that incorporates the intricate vortex dynamic effects, revealed in non-idealized experimental configurations. Magnetically anisotropic critical currents and flux stirring resistivities are straightforwardly included in three dimensional scenarios.

Starting from a variational form of our postulate, a numerical implementation for practical configurations is shown. In particular, several results are provided for the infinite strip geometry: voltage generation in multicomponent experiments, and magnetic relaxation towards the critical state under applied field and transport current. Explicitly, we show that for a given set of external conditions, the well established critical states may be utterly obtained as diffusive final profiles.

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1. Introduction

Over a half century, the critical state model [1] (CSM) has been an essential interpretative tool for the investigation of the magnetic properties of type-II superconductors. This accomplishment probably relies on a clear physical background together with an ostensible mathematical simplicity, that have enabled a huge number of utilizations both in material characterization as well as in more fundamental studies

In brief, the CSM postulates that the biased magnetic response of a type-II superconductor is defined by a series of equilibrium states characterized by a current density distribution $\mathbf{J}(\mathbf{r})$ that, in its simplest form, may be obtained by integration of Ampère's law given by $dH/dx = \pm J_c, 0$. J_c , the so-called critical current density is the single material parameter of the theory, and characterizes the balance equation between magnetic and intrinsic pinning forces: $\mathbf{J} \times \mathbf{B} = \mathbf{F}_p$. The transition between different configurations is assumed to take place instantaneously, which means that external field variations occur slowly enough as compared to the scale established by the material response ($\tau_{\text{ext}} \gg \tau_m$). Thus, although it is well known that magnetic flux penetration in the presence of pinning forces happens as an *avalanching process* [2] when the threshold condition is exceeded (i.e.: $J > J_c$), one argues that magnetic diffusion is so fast that the superconductor settles a negligible time in the intermediate resistive states.

In many instances, the CSM approximation is justified, but especially for the case of high T_c superconductors one can meet practical situations for which relaxation transients towards equilibrium configurations have to be considered. A very relevant case is the *fault current limiter* that, by construction operates driving the superconductor beyond the resistive transition [3, 4]. The precise knowledge of the magnetic diffusion processes that occur is obviously necessary for the design of such devices. In particular, the essential evaluation of energy losses may be seriously tampered if one neglects the effect of the sample becoming resistive. On the other hand, and more to the side of basic physics, one has to recall that, experimentally, the value of the critical parameter J_c is often obtained from transport measurements that are based on some threshold value for the voltage detected when the samples goes resistive (typically $1\mu\text{V}/\text{cm}$). In a recent work about the origin of the dissipation mechanisms that operate for superconductors in the vortex state, it has been argued that, although designed for obtaining the critical parameters, resistive measurements could be ineluctably recording properties of the resistive state [5]. This circumstance is relevant for HTS samples that display more gradual current voltage transitions as compared to the sharp behavior for conventional superconductors. Finally, it should also be mentioned that from the fundamental point of view, embedding the CSM approximation in a theory that allows to derive it as a limiting case is by itself a desirable objective.

In this work we put forward a minimal formulation that upgrades the critical state theory so as to enable the inclusion of *overcritical behavior*. The basic picture of our statement is as follows. Subject to an external action, the state of the superconductor will drift until new steady conditions are met. If allowed, the steady state will be an

equilibrium phase, characterized by a conventional *force balance* equation, with critical behavior for the current density. On the other hand, dissipative *steady states* and excursions from one critical state to the other will be described under a first order (linear) approximation for the underlying driving forces. Following the spirit of Bean's model, although describing phenomena whose ultimate nature would at least require a mesoscopic scale, we pursue a theory based on macroscopic variables. Thus, our basic assumption is that the superconductor behaves as a thermodynamic system, with the coarse-grained current density as the state variable, and postulate that dynamic properties may be predicted on the basis of three factors:

- (i) magnetic forces
- (i) material pinning forces
- (iii) irreversible thermal forces.

The latter group will allow to introduce the new physical effects related to dissipation. In order to avoid microscopical modeling, a power series expansion argumentation will be used for introducing such effects. Outstandingly, recalling universal properties as entropy increase and single valuedness of the physical observables, one can fairly determine the kind of models to be used, even in complex scenarios such as flux cutting environments or 3D modeling.

The article is organized as follows. Section 2 is devoted to deliver the formal details of our theory. First (Sec.2.1) we recall some mechanical concepts (fields, forces, Drude's model) that allow a rudimentary approach to the problem of dissipation in normal conducting systems. Then, (Sec.2.2) some basic thermodynamic background is introduced, with the aim of paving the way for the generalization to superconductors. As a central result of our work, in 2.2.2 we introduce the idea of *dissipation function* and entropy generation for type-II superconductors driven out of equilibrium. We will show that, concomitant with a complex structure for the equilibrium states, an acceptable theory of resistive losses has to fulfill some consistency requirements, that will be used to put restrictions on the possible material laws to be used ($\mathbf{E}(\mathbf{J})$ in particular). In order to ease the practical implementation, a variational form of our theory is issued. The second part of our paper (Sec.3) illustrates the application of the above concepts to practical situations. Mainly focused on the infinite strip geometry (quasi-1D configuration), we show that our basic equation allows to obtain the electromagnetic quantities either in equilibrium, steady states or during transient processes. It is explicitly shown that, for a given set of external conditions, relaxation eventually leads to the well known corresponding critical states. Excellent comparison with analytical results, when available, is displayed. The final section (Sec.4) summarizes our results and contains a brief discussion of possible applications and extensions of our work in the area of type-II superconductivity.

2. Resistive losses in type-II superconductors

2.1. Coarse grained modeling: fields, forces and $E(J)$ law

Let us start by recalling some details about the classical description of electrical conduction in normal metals, *viz.*, the Drude model. As we will see, this simple scheme may be illuminating for the issue of a minimal model of the resistive behavior of hard superconductors.

2.1.1. Normal metals

The simplest description of electric current in normal conductors (Drude model) is built from the classical dynamical equation of the charge carriers subject to both an external electric field and a phenomenological drag force \mathbf{F}_{drag} standing for the interaction with the molecular environment and other charges, i.e.:

$$m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} + \mathbf{F}_{\text{drag}}(\mathbf{v}_e). \quad (1)$$

Customarily, \mathbf{F}_{drag} is taken to first order, i.e.: $\mathbf{F}_{\text{drag}} \approx -(\mathbf{v}_e/\tau_{\text{tr}})$ and it is a characteristic of the material through the time constant τ_{tr} . It is important to recall that all the quantities in the above formulae have to be interpreted as mesoscopic averages both in spatial regions and time intervals. This entails smoothing (by randomizing) the macroscopically irrelevant fluctuations of the microscopic level. In a thermodynamic language, one speaks about a thermalized electron-lattice system. In spite of its simplicity, the model catches the basic behavior observed in an overwhelming amount of experiments.

It is of particular interest to observe that, unless for very high excitation frequencies (in the range of 1 THz for typical metals), a further simplification is still possible. A quasistatic approximation may be used, that implies to neglect the time derivative to the left hand side of Eq.(1). Then, the non-dispersive form of Ohm's law arises, i.e.: $\mathbf{J} = ne^2\tau_{\text{tr}}\mathbf{E} \equiv \sigma_0\mathbf{E}$ with n the number of conduction electrons per unit volume. We note that the electromagnetic behavior of conventional electric machines may be accurately predicted by combining this approximation for the material law with the so-called magnetoquasistatic limit of the Maxwell equations [6] for the macroscopic fields. Physically, as related to the smallness of τ_{tr} charge recombination processes within the conductor are neglected (i.e.: $\nabla \cdot \mathbf{J} = 0$). In practice, when studying the evolution induced by some process of the external excitation, this means that one starts by calculating the *dominant* magnetic fields from the diffusion equation, that for latter convenience will be expressed in terms of the material resistivity $\rho_0 \equiv 1/\sigma_0$

$$\left(\mu_0 \frac{\partial}{\partial t} - \rho_0 \nabla^2 \right) \mathbf{H} = 0 \quad (2)$$

Eventually, \mathbf{E} is obtained from Faraday's law. Notice that *stationary* solutions of this equation verify the condition $\nabla \times \mathbf{J} = 0$, that is to say: persistent current loops within the sample are excluded for a normal metal.

2.1.2. Hard superconductors

Microscopically, the nature of resistive losses in flux penetrated type-II superconductors is quite more complex than the scattering of normal electrons, which is behind the above summarized Drude's model. Nevertheless, one can take benefit of the conceptual aspects introduced, that may be straightforwardly applied at the phenomenological level. Thus, it is well known that under the action of a transport current and a perpendicular magnetic field, these materials experience a resistive transition, also characterized by a linear $\mathbf{E}(\mathbf{J})$ relation, now in terms of the so-called flux-flow resistivity ρ_{ff} . This behavior is well understood at the mesoscopic level. Abrikosov vortices are driven by the Lorentz-like force $\mathbf{J} \times \mathbf{B}$ and their normal cores contribute to the transport current as a normal channel. Then, in the absence of additional effects, one could formulate the behavior of the superconductor in the same terms described above for a normal metal, just replacing ρ_0 by ρ_{ff} .

However, the detrimental flux flow behavior has been successfully attacked by a number of pinning strategies. Basically, they rely on the idea of introducing a *restoring* force on the vortices, so as to keep them in equilibrium positions while a transport current flows along the system ($\mathbf{J} \times \mathbf{B} = \mathbf{F}_p$). As pinning forces are bounded ($F_p \leq F_{p,\text{max}}$), there will be a threshold value for the current density that can flow lossless, the so-called *critical current density* J_c . In this framework, the calculations leading to the evaluation of magnetostatic equilibrium properties may be rather simple. Thus, concentrated on these, Bean's postulate states that *flux penetrated regions will be characterized by the threshold (critical) conditions*, i.e.: $J = J_c$. Following this, in 1D problems one just has to integrate Ampère's law in the form $dH/dx = \pm J_c, 0$ supplied by specific boundary conditions for the magnetic field. Nevertheless, the investigation of the transient processes involving the appearance of resistivity may be rather complex, even for the simplest geometries. The reason is that now, effective *drag forces* only occur for currents circulating with a density beyond the critical value J_c . In other words, one has to deal with a material law of the kind $E = \rho(J)J$ where

$$\rho(J) = \begin{cases} 0 & \text{if } |J| \leq J_c \\ \rho_{\text{ff}} & \text{if } |J| > J_c \end{cases} \quad (3)$$

As a consequence of the non-linearity introduced by the piece-wise constant behavior of $\rho(J)$, one has to deal with noticeable difficulties, as for instance, the form taken by the diffusion equation (compare to Eq.(2))

$$\left(\mu_0 \frac{\partial}{\partial t} - \rho(\mathbf{J}) \nabla^2 \right) \mathbf{H} = (\nabla \times \mathbf{H}) \times \nabla \rho(\mathbf{J}) \quad (4)$$

Difficulties for solving this equation by analytical methods have been reported in Refs. ([3, 4]) that basically suggest to apply *ad hoc* approximations when justified by

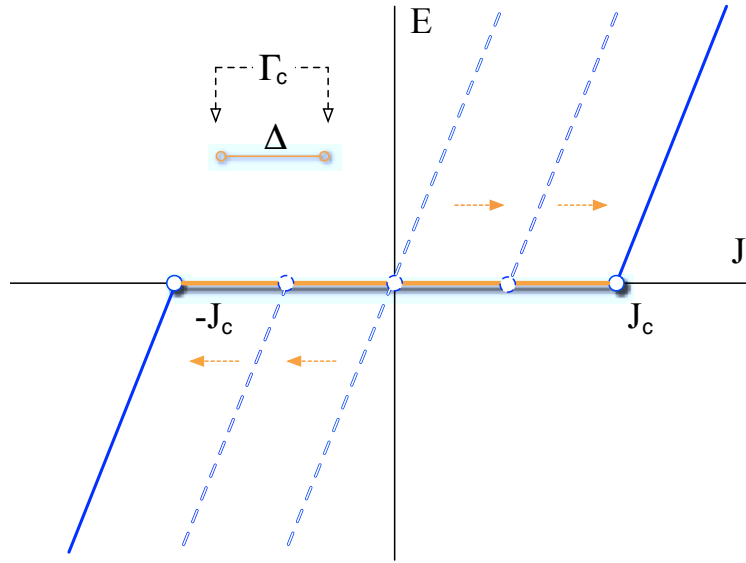


Figure 1. (Color online) Piecewise linear $E(J)$ law for a type-II superconductor. The material parameter J_c defines the boundary Γ_c of the lossless behavior (region $\Delta \equiv [-J_c, J_c]$). Ohm's law corresponds to $J_c = 0 \Rightarrow \Delta = \emptyset$.

the set of experimental data under consideration. On the other hand, we stress that, in fact, Bean's approximation means to avoid such equation under the following thoughts. If the actual material law in Eq.(3) is such that one can speak about a sharp transition, i.e.: ρ_{ff} takes elevated values as compared to the experimental parameter E/J , one can approximate $\rho_{\text{ff}} \rightarrow \infty$. Then, the diffusion time ($\tau_\rho \approx 1/\rho$, see τ_{tr} above) may be neglected and externally induced evolutions may be accurately described as a series of equilibrium *critical* states.

Fig.1 displays the main features of the above mentioned modeling for the case of 1D systems. Let us analyze the physical significance of such a material law.

- (i) Note first that superconducting lossless transport is allowed by the condition $E = 0$ within the region $\Delta \equiv [-J_c, J_c]$.
- (ii) Persistent shielding is also possible, subsequent to electromagnetic induction because E can go to zero and J take a local structure that allows $\nabla \times \mathbf{J} \neq 0$ when diffusion stops. In particular, this includes the critical state solutions with J taking the values $\pm J_c$ or 0 within the sample.
- (iii) One can visualize Bean's model as a limiting case of this material law when there is an arbitrarily high slope of $E(J)$ for $J > J_c$.
- (iv) The conceptual implications of Fig.1 may be straightforwardly generalized to higher dimensions. In particular, Δ may be a region of the \mathbf{J} -space and Γ_c its boundary. Thus, for the case of high- T_c superconductors it has been recently show [5] that a meaningful selection would be to take Δ as an elliptic region, within the plane defined by the components of \mathbf{J} parallel and perpendicular to the local magnetic

field.

- (v) The material law for a type-II superconductor depends not only on the intrinsic parameter ρ_{ff} , but is strongly affected by extrinsic quantities. In particular, the ambient magnetic field will play a prominent role for the definition of the region Δ . Thus, already in the simple 1D cases, the dependence $J_c(H)$ is usually an important concern. On the other hand, as mentioned above, the relative orientation of the vectors \mathbf{J} and \mathbf{H} is crucial for the material law in higher dimensions.

Performing calculations within the above described material law and its *natural* extensions to 3D is by no means a simple task. Below, we suggest a first approximation to the problem that relies on thermodynamic concepts applied either close to equilibrium or close to steady states. This will provide a useful framework for developing numerical tools that allow to analyze the resistive transition in applied superconductivity.

2.2. Thermodynamic background

Non-equilibrium thermodynamics is still an area under development. One of the main fields of interest, i.e.: the study of steady states will guide us for the analysis of electrical conduction. In fact, being concentrated on quasisteady systems, the extension to irreversible transient processes (magnetic diffusion) will be straightforward. As an example of what can be done, we recall [7] that based on the law of increasing entropy and also on Onsager reciprocal relations, one can show that in normal conductors, the conductivity tensor must be positive definite and symmetric. This generalizes 1D Ohm's law. Obviously, such properties could never be deduced from the pure electromechanical principles used in the previous paragraph. Thus, having the aim of a theory that generalizes Sec.2.1.2 for type-II superconductors, in the forthcoming, we present a minimal conceptual scenario that allows to host the main physical properties of these materials. As before, we will start introducing the main ideas by taking benefit of our knowledge of the simpler normal conductors.

2.2.1. Entropy and dissipation function in normal metals

Let us start by recalling some definitions. Being interested in local properties of the conductor, we consider the *mesoscopic* entropy function \mathcal{S} as an average that has smoothed the statistical microscopic fluctuations. In fact, the spatial dependence comes through the current density, that will be our *state variable*: $\mathcal{S}[\mathbf{J}(\mathbf{r})]$. Here, \mathbf{r} denotes the position of a region of mesoscopic size, and nonlocal correlations are neglected. In the absence of an external action the entropy reaches a maximum at the stable equilibrium point $\mathbf{J} = 0$, i.e., an overwhelming amount of all possible microstates corresponds to macrostates with $\mathbf{J} \simeq 0$, and then $\mathcal{S}(\mathbf{J}) \leq \mathcal{S}(\mathbf{J} = 0) = \mathcal{S}_{\text{eq}}$. Statistical fluctuations around such point become negligible with increasing size of the subsystem.

Out of equilibrium, but not far from it, the same statistical mechanism that suppresses fluctuations will operate. Thus, after a displacement caused by an external agent (electric field in our case), a thermodynamic “restoring force” \mathbf{F}_{drag} in Sec.2.1.1,

drives the system along increasing entropy until \mathcal{S}_{eq} is reached again. \mathbf{F}_{drag} is responsible for the energy losses that may be expressed in terms of the so-called dissipation function $\mathcal{F} = (1/2)\mathbf{F}_{\text{drag}} \cdot \mathbf{v}_e$, that measures the amount of heat generated per unit time. Macroscopically, we have

$$dQ = Td\mathcal{S} \equiv \mathcal{F}dt. \quad (5)$$

Hereafter, isothermal conditions will be assumed, which imply the relation $\dot{\mathcal{S}} = \mathcal{F}/T$. Now, thermodynamics enters through the so-called “principle of minimum entropy production” [8,9] that may be expressed as

The steady state of a system is that state in which the rate of entropy production has the minimum value consistent with the external constraints.

Notice that, based on this, the equilibrium state takes its natural place when there are no constraints on the system, because as a consequence of the second law, it reaches the maximum entropy, and thus the absolute minimum of entropy production, i.e.: zero.

In order to see how the above principle operates, let us assume that some external action (electric field) drives the system out of equilibrium. If the constraint is removed, a transient towards equilibrium (\mathcal{S}_{eq}) will occur. Around this point, the function \mathcal{F} will allow a quadratic expansion of the kind

$$\mathcal{F} \simeq \frac{1}{2} \sum_{ij} J_i \Omega^{ij} J_j. \quad (6)$$

with Ω^{ij} the components of a symmetric positive definite tensor. It is apparent that, in the absence of constraints, minimizing \mathcal{F} leads to the condition $\mathbf{J} = 0$, and that the thermodynamic field driving the system towards equilibrium may be obtained as $\mathbf{E}_{\text{ther}} = -\nabla_{\mathbf{J}}\mathcal{F} = -\Omega\mathbf{J}$. Its lines of force (maximal slope of \mathcal{F}) in the \mathbf{J} -space correspond to the most probable macroscopic evolution (faster increase of entropy). In steady situations, this field *balances* the applied electric field that verifies Ohm’s law ($-\mathbf{E}_{\text{ther}} = \mathbf{E} = \Omega\mathbf{J}$).

On the other hand, steady states out of equilibrium will be characterized by a constrained minimization problem, in which \mathcal{F} has to be *augmented* by some Lagrange multiplier. For instance, as indicated by Landau [7], the steady current distribution within a normal conductor may be obtained by minimizing the volume integral of $\mathcal{F} + \lambda \nabla \cdot \mathbf{J}$ as corresponds to the stationarity condition $\nabla \cdot \mathbf{J} = 0$.

As a main conclusion from the above paragraphs, we resolve that the behavior of a normal conductor close to equilibrium may be formulated in terms of a quadratic tensor Ω^{ij} (resistivity) that is positive definite and symmetric as dictated by thermodynamics. This obviously generalizes the one dimensional results in Sec.2.1.1. An additional observation is that just recalling the definition of the gradient function one can show that the electric field lines will be perpendicular to the constant level sets of the dissipation function \mathcal{F} . Finally, variational principles allowing to afford complicated calculations as those related to the diffusion towards the *equilibrium point* ($\mathbf{J} = 0$) or characterizing steady states may be stated in this realm.

Below, we show that these ideas may be exported to the case of type-II superconductors. However, as related to the generalization of the one dimensional $E(J)$ law in Fig.1, some further analysis is required.

2.2.2. Entropy and dissipation function in type-II superconductors

In the light of Sec.2.1.2 the concept of thermodynamic equilibrium has to be reconsidered for pinned superconductors. Thus, already for the simplest one dimensional configurations, one finds a vanishing drag force within a segment (region Δ) that becomes a surface, or even a volume in more complex scenarios. Then, the entropy of the system will be maximal within a full set of points $\mathbf{J}_c \in \Delta$ and the determination of the state from thermodynamic arguments seems flawed by ambiguity. Against this, one could argue that, in fact, either static or stationary configurations always occur subsequent to some diffusion process accompanied by drag forces (electric fields) and that “integration” along some path within the \mathbf{J} -space that connects the overcritical region and the eventual critical point \mathbf{J}_c can be performed. However, the following mathematical quiz arises: how does one expand a function \mathcal{F} towards the resistive behavior when a “starting region”, instead of a starting point, is given?

Once more, a very general argument, uniqueness, will help. Let us consider the kind of elliptical region mentioned before, that is physically meaningful as related to dissipation mechanisms in either J_{\parallel} or J_{\perp} relative to the local magnetic field orientation. As depicted in Fig.2, irreversible energy dissipation corresponding to some overcritical point $\mathbf{J} \notin \Delta$ may be uniquely quantified by means of d_{Γ} , i.e.: the minimum distance from the point \mathbf{J} to the boundary Γ_c . Geometrically, this means that expansion is done perpendicular to Γ_c . Analytically, one has to find some expression that allows to obtain the critical point \mathbf{J}_c for each value of \mathbf{J} . In the case of the elliptic region considered here, this entails to solve a quartic equation. Of special mention is that meaningful cases as the isotropic model (Δ is a circle) and the Double Critical State Model (Δ is a rectangle) produce trivial conditions for the determination of the point \mathbf{J}_c .

From the physical point of view, the above mathematical conditions imply that for small perturbations around equilibrium, the induced electric fields will keep perpendicular to the levels of constant dissipation \mathcal{F} . This extends our previous result [10] that just “on surface” (critical state) \mathbf{E} is normal to the boundary of the region Δ . At least for small perturbations, a consistent theory that assumes the existence of a critical region with boundary Γ_c may be obtained by using $\mathbf{E} \perp \Gamma_c$ and prolong this condition towards the resistive state. Below, we will be more explicit on the mathematical aspects related to the material law.

2.2.3. Thermodynamically admissible $\mathbf{E}(\mathbf{J})$ laws: mathematical issues

The dissipation function \mathcal{F} must be defined, positive and obviously single valued, outside the region Δ of (sub)critical current densities, in the \mathbf{J} vector space

$$\mathcal{F} : \mathbb{R}^3/\Delta \rightarrow \mathbb{R}^+ \quad \mathcal{F}(\mathbf{J}) \geq 0 \quad (7)$$

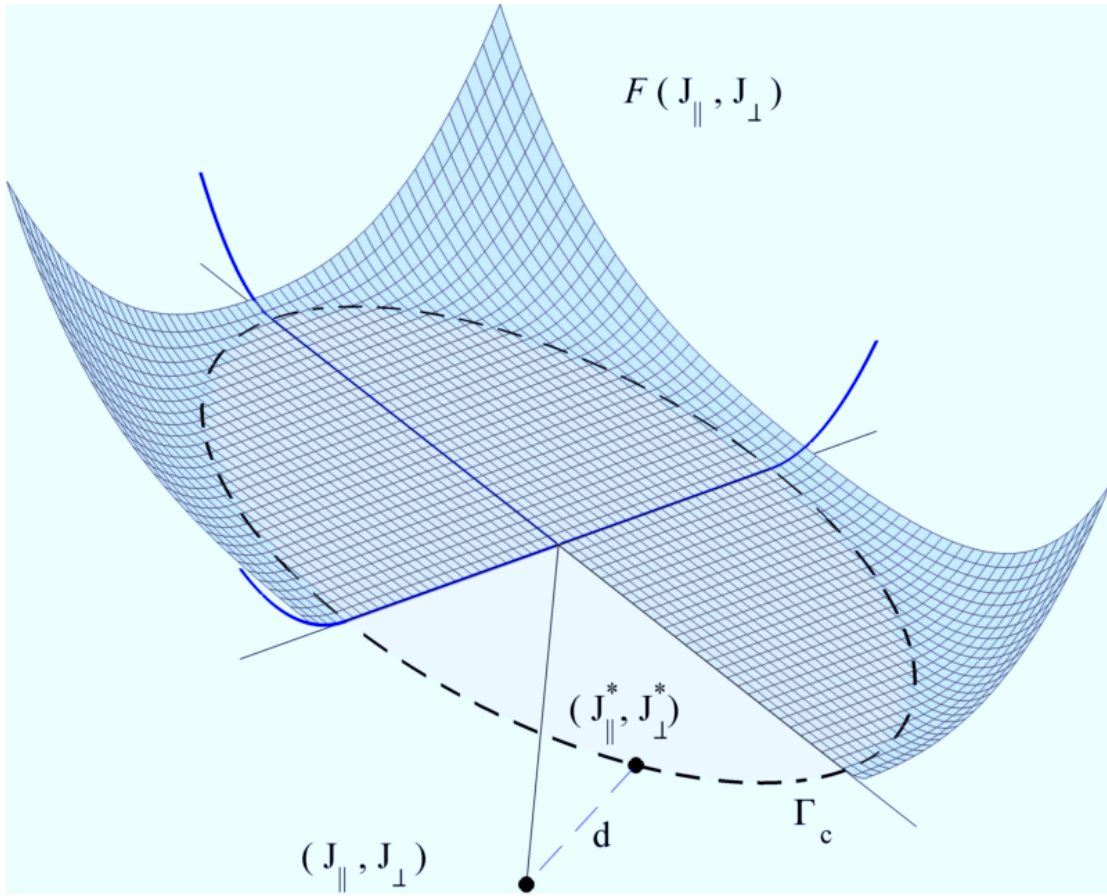


Figure 2. (Color online) Dissipation function for a type-II superconductor close to the critical state. For a given value of the current density vector $(J_{\parallel}, J_{\perp})$, \mathcal{F} is defined through the distance d to the critical region boundary Γ_c . The related critical point $\mathbf{J}_c \equiv (\mathbf{J}_{\parallel}^*, \mathbf{J}_{\perp}^*)$ is also shown.

A Taylor expansion around the boundary of critical currents Γ_c will encode the main properties of the electromagnetic behavior close beyond the critical state, i.e., the $\mathbf{E}(\mathbf{J})$ constitutive law.

To start with, we will consider spatially isotropic samples. Thus, the surface Γ_c will be axially symmetric around the local \mathbf{H} direction, as well as symmetric under reflection through the plane normal to \mathbf{H} . Δ is also supposed convex. With this symmetry, the local electric field takes value in the plane defined by \mathbf{H} and \mathbf{J} . Hereafter, we restrict the analysis to this plane, and use cartesian coordinates (J_1, J_2) , (E_1, E_2) and $(H_1, 0)$. Γ_c is then a closed curve surrounding the convex region of subcritical currents. In polar coordinates $(J_1, J_2) = (J \cos(\theta), J \sin(\theta))$, and Γ_c is determined by a function $J_c(\theta)$.

Let us see how the concept of perpendicular expansion introduced above arises. In principle, there is no particular point at Γ_c from which to perform the Taylor expansion.

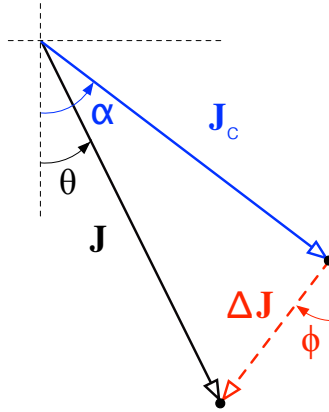


Figure 3. (Color online) Schematics of the current density vectors in the over-critical state problem (as in Fig.2). \mathbf{J} stands for the actual current density, \mathbf{J}_c for the nearest critical current density vector, and $\Delta\mathbf{J}$ for their difference. Related angles are also defined.

Let us then consider an arbitrary $\mathbf{J}_c^{(1)} \in \Gamma_c$, and denote by $T(\mathbf{J}; \mathbf{J}_c^{(1)})$ the Taylor expansion of $\mathcal{F} \simeq T(\mathbf{J}; \mathbf{J}_c^{(1)})$. It is reasonable to take a point $\mathbf{J}_c^{(1)}$ close to the value \mathbf{J} of interest, in particular, nearest point to \mathbf{J} on the critical surface. However, a well behaved function \mathcal{F} should admit compatible approximations from nearby points, say, $\mathcal{F} \simeq T(\mathbf{J}; \mathbf{J}_c^{(2)})$. Therefore

$$T(\mathbf{J}_0; \mathbf{J}_c^{(1)}) \simeq T(\mathbf{J}_0; \mathbf{J}_c^{(2)}) \simeq T(\mathbf{J}_0; \mathbf{J}_c(\theta)) \quad \Rightarrow \quad \partial_\theta T = 0 \quad (8)$$

i.e., when fixing a value \mathbf{J}_0 , the derivative of $T(\mathbf{J}_0; \mathbf{J}_c)$ along the curve Γ_c vanishes.

Now, for our purposes it suffices to include second and third order terms

$$\begin{aligned} T(\mathbf{J}; \mathbf{J}_c^{(1)}) = & \frac{1}{2}[\rho_{11}(J_1 - J_{1c}^{(1)})^2 + \rho_{22}(J_2 - J_{2c}^{(1)})^2 + \\ & + 2\rho_{12}(J_1 - J_{1c}^{(1)})(J_2 - J_{2c}^{(1)})] + \\ & + \frac{1}{6}[\gamma_{111}(J_1 - J_{1c}^{(1)})^3 + 3\gamma_{112}(J_1 - J_{1c}^{(1)})^2(J_2 - J_{2c}^{(1)}) + \\ & + 3\gamma_{122}(J_1 - J_{1c}^{(1)})(J_2 - J_{2c}^{(1)})^2 + \gamma_{222}(J_2 - J_{2c}^{(1)})^3] \end{aligned}$$

with

$$\rho_{ij} = \frac{\partial^2 \mathcal{F}}{\partial J_i \partial J_j}(\mathbf{J}_c^{(1)}) \quad \gamma_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial J_i \partial J_j \partial J_k}(\mathbf{J}_c^{(1)}). \quad (9)$$

From the condition $\partial_\theta T = 0$ we get, to first order

$$(\rho_{11}, \rho_{12}) \cdot \mathbf{T}(\theta) = 0 \quad (\rho_{12}, \rho_{22}) \cdot \mathbf{T}(\theta) = 0 \quad (10)$$

so that $\mathbf{T} = \partial_\theta(J_c(\theta) \cos(\theta), J_c(\theta) \sin(\theta))$, the tangent vector to Γ_c , belongs to the kernel of the resistivity tensor

$$\Omega = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{pmatrix} \quad (11)$$

On the other hand, to second order

$$\begin{aligned}(\gamma_{111}, \gamma_{112}) \cdot \mathbf{T}(\theta) &= -\partial_\theta \rho_{11} \\(\gamma_{112}, \gamma_{122}) \cdot \mathbf{T}(\theta) &= -\partial_\theta \rho_{12} \\(\gamma_{122}, \gamma_{222}) \cdot \mathbf{T}(\theta) &= -\partial_\theta \rho_{22}\end{aligned}$$

Note that for the strict second order Taylor polynomial ($\gamma_{ijk} = 0$), the compatibility condition implies that Ω is constant, but this is no longer true when third order terms are considered.

Now, if we use the vector basis $\{\hat{\mathbf{t}}, \hat{\mathbf{n}}\}$, i.e.: unit tangent and normal vectors at a given point of Γ_c , the tensor Ω becomes

$$\Omega = \begin{pmatrix} 0 & 0 \\ 0 & \rho_n \end{pmatrix} \quad (12)$$

Thus, the Hessian of \mathcal{F} is positive semidefinite, with $\hat{\mathbf{t}}$ null eigenvector, and ρ_n the eigenvalue of $\hat{\mathbf{n}}$.

Remarkably, one can maintain the simplicity of the second order approach while allowing anisotropic $\Omega(\theta)$, by performing the Taylor expansion as follows; from each point $\mathbf{J}_c \in \Gamma_c$ we compute the Taylor expansion “exclusively” along the normal halfline $\mathbf{J}_c + \Delta J \hat{\mathbf{n}}$. In such a way, and taking into account the convexity of the base curve, there is a univocal correspondence of each \mathbf{J} with a particular Taylor expansion, and the whole region \mathbb{R}^2/Δ_c is covered. The second order approach to the dissipation function becomes

$$\mathcal{F}(\mathbf{J}) = \frac{1}{2} \rho(\alpha) (\Delta J)^2 \quad \mathbf{J} = \mathbf{J}_c(\alpha) + \Delta J \hat{\mathbf{n}}(\alpha) \quad (13)$$

The correspondence $\mathbf{J} \equiv (J, \theta) \leftrightarrow (\Delta J, \alpha)$ is the geometric condition of nearest point of the curve Γ_c to \mathbf{J} , and it obviously depends on the specific $J_c(\alpha)$ function. Note the distinction between θ and α , the angular coordinates of \mathbf{J} and \mathbf{J}_c (Fig.3).

Eventually, the electric field ($\mathbf{E} = \nabla_{\mathbf{J}} \mathcal{F}$) is given by the polar coordinate expression

$$\mathbf{E} = \partial_J \mathcal{F} \hat{\mathbf{J}} + \frac{1}{J} \partial_\theta \mathcal{F} \hat{\theta} = E_{\parallel} \hat{\mathbf{J}} + E_{\perp} \hat{\theta}. \quad (14)$$

For practical purposes, an intrinsic coordinate system representation will be useful. Thus, a better adapted expression in polar-like coordinates $\{\Delta J, \phi\}$, with ΔJ the distance of \mathbf{J} to Γ_c , and ϕ the angle between $\Delta \mathbf{J}$ and the \mathbf{H} axis reads

$$\mathbf{E} = \partial_{\Delta J} \mathcal{F} \hat{\Delta \mathbf{J}} + \frac{1}{l_\phi} \partial_\phi \mathcal{F} \hat{\phi}. \quad (15)$$

Here l_ϕ is the length of $\partial_\phi \mathbf{J} = \partial_\phi \mathbf{J}_c + \Delta J \partial_\phi \hat{\mathbf{n}}$, and $\hat{\Delta \mathbf{J}} \equiv \hat{\mathbf{n}}$. Then, one has

$$\mathbf{E} = \rho(\phi) (\Delta J) \hat{\mathbf{n}} + \frac{1}{2l_\phi} \partial_\phi \rho (\Delta J)^2 \hat{\phi} \quad (16)$$

Notice that, in case of a constant ρ , the electric field is always parallel to $\hat{\mathbf{n}}$. However, as it will be analyzed below, in connection with voltage-current experiments, the ratio E_{\parallel} and E_{\perp} changes with the separation of the working point and the critical region $\Delta \mathbf{J} = \mathbf{J} - \mathbf{J}_c$ (see section 3.2).

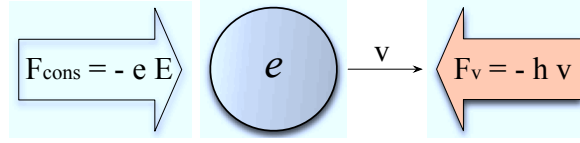


Figure 4. Schematics of a simple irreversible system: a charge carrier subjected to the action of an electric field and a simultaneous viscous force.

2.2.4. Variational formulation of the conduction problem

It has been argued that classical mechanics and its methods do not provide a complete framework for the analysis of electrical conduction. However, having exploited the consequences of the second law of thermodynamics, one can reconsider the problem. In particular, below we show that by including the concept of dissipation function \mathcal{F} one can issue a variational formulation for the magnetic diffusion problem between steady conduction states. In this case, a unified description that encompasses normal conductors and type-II superconductors is presented.

Just with illuminating purpose we start by considering a 1D problem with a charged particle subject to an electric field E with associated potential energy U . Recall that the Lagrangian formulation of Hamilton's principle is as follows

$$S \equiv \int L(x, v) dt = \int \left[\frac{mv^2}{2} - U(x) \right] dt$$

$$\text{Min } S \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0 \Leftrightarrow m \frac{dv}{dt} = - \frac{\partial U}{\partial x} \equiv F_{cons}$$

where F_{cons} stands for the *conservative* force $F_{cons} = qE$. Consider now that a viscous drag acts on the particle (as depicted in Fig.4 for the case of an electron). Notice that a minimum principle leading to the sound equations of motion can still be formulated for a modified Lagrangian as shown below

$$\hat{S} \equiv \int \hat{L}(x, v, t) dt \quad \text{with} \quad \hat{L} \equiv L + \frac{1}{2} h v^2 t$$

$$\text{Min } \hat{S} \Rightarrow \frac{d}{dt} \frac{\partial \hat{L}}{\partial v} - \frac{\partial \hat{L}}{\partial x} = 0 \Leftrightarrow m \frac{dv}{dt} \simeq - \frac{\partial U}{\partial x} - h v = F_{cons} + F_v$$

What has been done in deriving such formulation is to neglect variations of the viscous force F_v within the interval of time considered. Then, the suggested approximation will be valid if minimization is applied “iteratively” with intervals of duration much less than the characteristic time $\tau \equiv h/m$. This relates to the so-called *Adiabatic Hypothesis* used in other physical disciplines:

Energy, though not conserved, varies slowly according to some parameter and one may assume a kind of isolated system within each interval.

In passing, notice that, upon neglecting the term $\Delta F_v \Delta x$ the average energy within a

given interval is

$$\mathcal{E} = v \frac{\partial \hat{L}}{\partial v} - \hat{L} \simeq \frac{mv^2}{2} + \frac{1}{2} F_v v \Delta t \quad (17)$$

This relation, that includes a contribution accounting for the dissipated energy ($\langle F_v v \rangle$ or \mathcal{F} in more general terms), is a means of understanding the principle of *minimum entropy production* in quasisteady processes. Within the adiabatic hypothesis one can use the second law of thermodynamics in the form: *for a closed system with fixed entropy, the total energy is minimized at equilibrium*.

Let us now see how the above arguments may be exported to the problem of electrical conduction. Within the quasisteady approximation, the incremental time-averaged field Lagrangian for conducting materials reads

$$\langle \hat{L} \rangle = \frac{\mu_0}{2} \int_{\mathbb{R}^3} \|\mathbf{H}_{n+1} - \mathbf{H}_n\|^2 dV + \int_{\text{Vol}} \Delta t \mathcal{F} dV \equiv \int_{\mathbb{R}^3} \hat{\mathcal{L}} dV \quad (18)$$

with \mathcal{F} the dissipation function introduced in Sec.2.2.

For the case of a normal conductor, it is relatively simple to show that the Euler-Lagrange equations lead to the desired diffusion equation (Eq.(2)). Thus, if one assumes a diagonal resistivity matrix, i.e.: $\Omega^{ij} = \rho_0 \delta^{ij}$ it follows

$$\begin{aligned} \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{H}_{n+1}} &= \sum_j \frac{\partial}{\partial x_j} \frac{\partial \hat{\mathcal{L}}}{\partial (\partial \mathbf{H}_{n+1} / \partial x_j)} \\ &\Downarrow \\ \mu_0 \frac{\mathbf{H}_{n+1} - \mathbf{H}_n}{\Delta t} &= -\rho_0 \nabla \times \nabla \times \mathbf{H}_{n+1} = \rho_0 \nabla^2 \mathbf{H}_{n+1} \end{aligned} \quad (19)$$

i.e.: the time discretized form of Eq.(2).

Outstandingly, the interest of Eq.(18) is that it gives way to the possibility of applying direct numerical minimization methods. In particular, this allows to deal with non-simple forms of \mathcal{F} as required for the investigation of type-II superconductors (recall Fig.2).

From the mathematical point of view, one can distinguish two kinds of problems as related to the solution of Eq.(18): (a) strictly variational structures, when either the first or the second term may be neglected, and (b) a quasi-variational structure when both are relevant [11]. Physically, one would speak about:

- (i) Equilibrium processes, leading to the critical state (dissipation function is not relevant).
- (ii) Steady states within a dissipative regime (magnetic inertia $\Delta \mathbf{H}$ is neglected, or external sources are fixed).
- (iii) Quasisteady evolutions when both inertia and dissipation have to be included (system diffuses towards the critical state).

The rest of this paper will be focused on several examples in which the properties of type-II superconductors close to equilibrium (critical state) are calculated based on the theory issued in this section. The presentation will be organized according to the three cases described above.

3. Numerical applications

As it was explained in Sec.2, the core of our theoretical proposal is the existence of a dissipation function \mathcal{F} that one may write down as a quadratic expression of the macroscopic current density vector components. Recall that both from the mathematical point of view, and also as concerns the physical background, \mathcal{F} should be defined as from certain distance d . Actually, d is a measure of the separation between the operation point (given by the value of the current density \mathbf{J}) and a certain equilibrium value \mathbf{J}_c that depends on \mathbf{J} itself. The specific form of the function \mathcal{F} depends on the critical current region boundary Γ_c , as we have explicitly shown in Fig.2 for the case of an elliptic behavior.

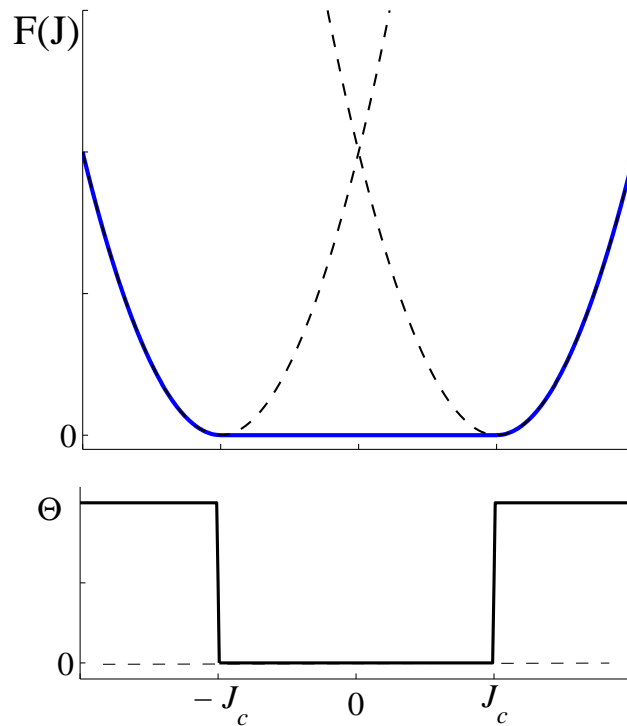


Figure 5. (Color online) Detail of a one-dimensional dissipation function for type-II superconductors. \mathcal{F} is built upon multiplying two parabolae by the step functions depicted below and adding them.

In this section, we present a number of examples in which the above ideas have been applied to various practical configurations. Let us start by writing down some useful

equations. Just for clarity, our statements refer to the region in Fig.2, but generalization is apparent.

- Notice that one may classify a given value of \mathbf{J} (in components $(J_{\parallel}, J_{\perp})$) as lying inside Γ_c or not, by calculating the sign of the quantity

$$\delta_{\Gamma}(\mathbf{J}) = \left(\frac{J_{\parallel}}{J_{c\parallel}} \right)^2 + \left(\frac{J_{\perp}}{J_{c\perp}} \right)^2 - 1, \quad (20)$$

i.e.: $\delta_{\Gamma}(\mathbf{J})$ is either negative or positive when \mathbf{J} is either within or beyond the contour of Γ_c (and null for the *critical* values $\mathbf{J}_c \in \Gamma_c$).

- The distance function for a given point $(J_{\parallel}, J_{\perp})$, also involving its image $\mathbf{J}_c \equiv (J_{\parallel}^*, J_{\perp}^*)$ is given by

$$d_{\Gamma}^2 = (J_{\parallel} - J_{\parallel}^*)^2 + (J_{\perp} - J_{\perp}^*)^2. \quad (21)$$

Here, J_{\parallel}^* and J_{\perp}^* have to be determined for the actual critical current law (region) under consideration. In the case of an elliptic model, the criterion of minimum distance between $(J_{\parallel}, J_{\perp})$ and the boundary leads to solving a quartic equation.

- For a given value $(J_{\parallel}, J_{\perp})$, the dissipation function $\mathcal{F}(\mathbf{J})$ may be written as

$$\mathcal{F} = \frac{1}{2} \rho \Theta_{\Gamma}(\mathbf{J}) d_{\Gamma}^2(\mathbf{J}) \quad (22)$$

with ρ the material resistivity, and d_{Γ} the distance function referred above. Θ_{Γ} stands for a step function whose value is zero within Γ_c and one over the outside. A useful representation is

$$\Theta_{\Gamma}(\mathbf{J}) = \frac{1 + \tanh[k\delta_{\Gamma}(\mathbf{J})]}{2}, \quad k \gg 1. \quad (23)$$

Recall that for the rather usual experimental configuration in which $\mathbf{J} = \mathbf{J}_{\perp}$ (the components of \mathbf{J} parallel to the magnetic field are zero), the above formulation may be cast as follows (mind the lines in Fig.2):

$$\begin{aligned} \delta_{\Gamma} &= (J_{\perp}/J_{c\perp})^2 - 1 \\ J_{\perp}^* &= \begin{cases} J_{c\perp} & \text{if } J_{\perp} > J_{c\perp} \\ -J_{c\perp} & \text{if } J_{\perp} < -J_{c\perp} \end{cases} \\ d_{\Gamma}^2 &= \begin{cases} (J_{\perp} - J_{c\perp})^2 & \text{if } J_{\perp} > J_{c\perp} \\ (J_{\perp} + J_{c\perp})^2 & \text{if } J_{\perp} < -J_{c\perp} \end{cases} \end{aligned} \quad (24)$$

These relations mean that, for a vast number of experimental setups, \mathcal{F} may be built by composing two parabolae and a step function as illustrated in Fig.5. The examples supplied below are based on this assumption. For the readers' sake, we eventually provide a useful discrete form of Eq.(18) that is based upon the transformation of the electromagnetic problem in terms of potentials [12] and on identifying the set of elementary circuits related to the problem's symmetry, viz.

$$\underbrace{\frac{\mu_0}{2} \int_{\mathbb{R}^3} \|\mathbf{H}_{n+1} - \mathbf{H}_n\|^2 dV}_{\frac{1}{2} \sum_{i,j} I_i M_{ij} I_j - \sum_{i,j} \tilde{I}_i M_{ij} I_j + \sum_i I_i (A^e - \tilde{A}^e)} + \underbrace{\int_{\text{Vol}} \Delta t \mathcal{F} dV}_{\frac{1}{2} \Delta t \sum_i R_i \Theta(\pm I_{c_i}) (I_i \mp I_{c_i})^2} \quad (25)$$

Here A^e means the vector potential component related to the applied magnetic field, M_{ij} denotes the inductance between elementary currents I_i, I_j flowing along the specific circuits of the problem, and the tilded quantities concern previous time layer. Δt is the incremental time of our calculation, R_i stands for the incremental resistance of the i -th circuit and I_{c_i} its critical current. Application to different problems will imply different expressions for the matrix elements M_{ij} , the applied vector potential, the resistance and the critical currents.

Just for completeness, we mention that other possibilities for the dissipation function may be explored as it will be shown below. In fact, the customary power-law expression for the $E(J)$ law has also been investigated through the related dissipation function

$$\int_{\text{Vol}} \Delta t \mathcal{F} dV \rightarrow F_0 \Delta t \sum_i \left(\frac{I_i}{I_{c_i}} \right)^m \quad (26)$$

3.1. Equilibrium: the critical states

To start with, we present an application of Eq.(25) in a strictly “critical” situation, i.e.: the dissipation term is not included explicitly, but replaced by the constraint $J \leq J_c$ in the minimization of the first term. Physically, this corresponds to the conventional critical state problems in which overcritical excursions are neglected. We have chosen a case of interest, that has been solved analytically, thus allowing a straightforward comparison to our numerical results. As related to the tape geometry and also in connection to a big number of experiments, we apply our method to the circular disk geometry. According to Mikheenko and Kuzovlev [13], the sheet current density distribution in a superconducting disk subject to a uniform perpendicular field of intensity H_0 is given by

$$J(r) = \begin{cases} \frac{2J_c}{\pi} \tan^{-1} \left[\frac{(r/R) \sqrt{R^2 - a^2}}{\sqrt{a^2 - r^2}} \right] & r \leq a \\ J_c & a \leq r \leq R, \end{cases} \quad (27)$$

with $a \equiv R / \cosh(2H_0/J_c d)$. d stands for the thickness of the disk and R for its radius. Here, J denotes the so-called sheet current that averages the current density over the thickness.

Fig.6 displays the comparison of our results and those obtained from the above equation. We stress the fact that numerics produce the correct solution, even for situations in which the variable does not reach the true critical value. Recall that,

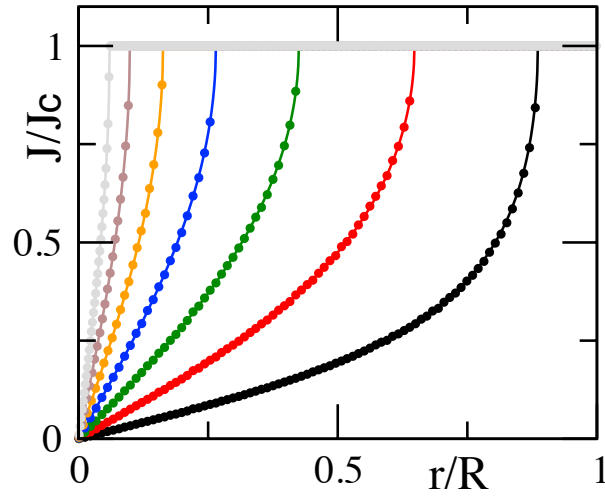


Figure 6. (Color online) (Sheet) current density profiles in the equilibrium (critical) states that are induced in a superconducting disk by an increasing magnetic field along its axis. Lines correspond to the analytical solution in Ref. [13] (see text) and symbols to our numerical results.

physically, the strict inequality ($J < J_c$) relates to the underlying average over the thickness, when a non-penetrated core still exists.

3.2. Steady states: the voltage criterion

Below, we concentrate on the opposite limit of the general conduction problem with type-II superconductors. The steady situation in which a dissipative state is maintained by the external action (current source) will be analyzed. Thus, based on the considerations introduced in Sec.2.2.3 for the law $\mathbf{E} = \nabla_{\mathbf{J}}\mathcal{F}$, we concentrate on recently reported voltage-current experiments [5], specifically designed to investigate the complex $\mathbf{E}(\mathbf{J})$ law and underlying critical boundary Γ_c in High- T_c superconductors. Owing to the smart design of the experiment, a direct analytical study is allowed.

First, we recall that the results in Ref. [5] show that the elliptic model for Γ_c gives an excellent fit to measurements. The authors have also obtained the polar angular dependence (see Fig.3 for definitions) of the angle between \mathbf{E} and \mathbf{J} , say $\beta(\theta)$. In their experimental configuration $\hat{\mathbf{J}}$ is fixed along the film direction, while $\hat{\mathbf{B}}$ is applied at different angles θ . The criterion for reaching and exceeding the critical state comes from a fixed value of \mathbf{E} parallel to the direction of \mathbf{J} ($E_{\parallel} \equiv \mathbf{E} \cdot \hat{\mathbf{J}} = E_0$).

From the elliptical curve of critical currents

$$\left(\frac{J_{\parallel}}{J_{c\parallel}}\right)^2 + \left(\frac{J_{\perp}}{J_{c\perp}}\right)^2 = 1 \quad (28)$$

we get the parametric representation

$$J_c(\alpha)^2 \left[\left(\frac{\cos(\alpha)}{J_{c\parallel}} \right)^2 + \left(\frac{\sin(\alpha)}{J_{c\perp}} \right)^2 \right] = 1 \quad (29)$$

in terms of which the normal vector is given by $(\cos(\alpha)/J_{c\parallel}^2, \sin(\alpha)/J_{c\perp}^2)$. Recalling the notation introduced in Sec.2.2.3, $\hat{\mathbf{n}} = (\cos(\phi), \sin(\phi))$, we have $\gamma^2 \tan(\phi) = \tan(\alpha)$, with the anisotropy ratio $\gamma = J_{c\perp}/J_{c\parallel}$.

Now, the ratio E_{\perp}/E_{\parallel} for small dissipation ($\Delta J \rightarrow 0$) is basically that of the electric field components at Γ_c , where $\mathbf{E} \perp \Gamma_c$ ($\beta = \phi - \theta$), i.e.:

$$E_{\perp}/E_{\parallel} = \tan(\phi - \theta) = \frac{\tan(\phi) - \tan(\theta)}{1 + \tan(\phi) \tan(\theta)} = \frac{\tan(\theta)(1 - \gamma^2)}{\gamma^2 + \tan^2(\theta)} \equiv \Phi_c(\gamma, \theta) \quad (30)$$

because in this limit $\alpha = \theta$.

In order to quantify the influence of the dissipation parameters in the electric field ratio, we proceed by considering a first order correction in terms of ΔJ . In the simplest case of constant ρ (see Sec.2.2.3) this maintains $\mathbf{E} \parallel \hat{\mathbf{n}}$, but now $\alpha \neq \theta$. From the trigonometric relations in the triangle of sides \mathbf{J} , \mathbf{J}_c and $\Delta \mathbf{J}$ and the condition $E_{\parallel} = E_0$ we get $\theta - \alpha \simeq \Phi_c E_0 / (\rho J_c)$, and then

$$\begin{aligned} E_{\perp}/E_{\parallel} &\approx \Phi_c \left[1 - \delta \gamma^2 (1 + \Phi_c^2) \frac{(1 + \tan^2(\theta))^{1/2} (\gamma^2 + \tan^2(\theta))^{1/2}}{\gamma^4 + \tan^2(\theta)} \right] \\ &\equiv \Phi_{\Delta J}(\gamma, \theta, \delta) \end{aligned} \quad (31)$$

where $\delta = E_0 / (\rho J_{c\perp})$ is the (very small) parameter in the expansion.

Eventually, we consider the most general case in which anisotropic resistivity is allowed. Let us assume the form $\rho = \rho_{\parallel} \cos^2(\theta) + \rho_{\perp} \sin^2(\theta)$, and $r^2 \equiv \rho_{\parallel} / \rho_{\perp}$ the anisotropy ratio. Starting with Eq.(16) and after some algebra we get

$$E_{\perp}/E_{\parallel} \approx \Phi_{\Delta J} + \delta \Phi_r (\Phi_{\Delta J}^2 - 1) \sqrt{1 + \Phi_c^2} \frac{(\gamma^2 + \tan^2(\theta))^{3/2}}{(r^2 + \tan^2(\theta))(\gamma^4 + \tan^2(\theta))^{1/2}} \quad (32)$$

with $\Phi_r \equiv (1 - r^2) \tan(\theta) / (r^2 + \tan^2(\theta))$.

Notice that the theory includes two parameters, δ and r^2 , that may be used to fit the experiment, and ultimately to recover the resistivity coefficients.

3.3. Transient behavior: relaxation towards the critical state

Finally, based on the application of Eq.(25), we put forward several cases that explicitly show the diffusion of electromagnetic fields in a type-II superconductor. We will concentrate on the tape geometry (Fig.7) and analyze relaxation towards the critical state, when either transport current or magnetic field steps are applied. By considering different values of the bias characteristic period τ_0 as compared to the material characteristic value τ_{ρ} , we will display different conditions in which critical

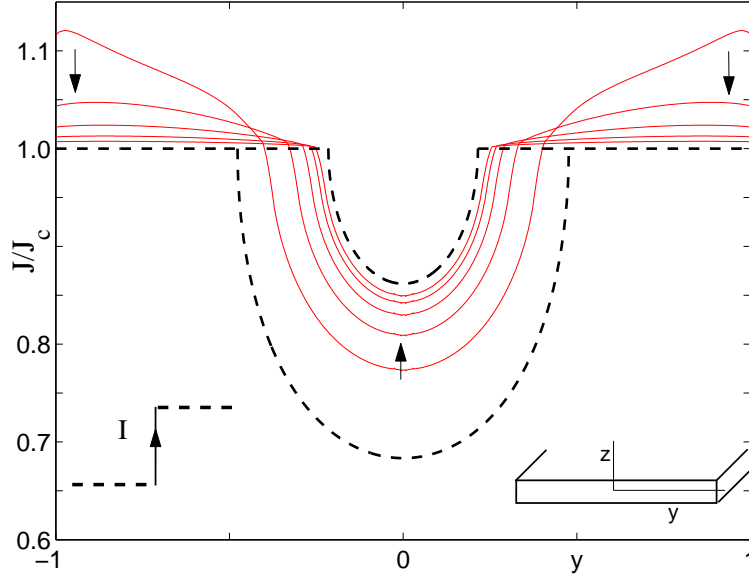


Figure 7. (Color online) Illustration of the diffusion processes simulated in this work. Here, a step is applied to the transport current I along the X -axis of a thin strip (as shown in the insets). The circulating sheet current $J(y)$ increases over the initial critical state profile (lower dashes) and then “relaxes” towards the subsequent critical state (upper dashes).

states are (or aren’t) realized after relaxation. The conditions studied may be applied to establish validity of the CSM approximation in AC experiments.

The general features of our simulations are shown in Fig.7. Notice that we obtain the current density profiles that occur between two equilibrium states when a step of transport current is introduced. Mainly, we have studied the influence of the ratio τ_0/τ_ρ on the diffusion process within the quadratic dissipation function framework developed in the previous sections. This relates to the piecewise linear behavior of the $E(J)$ law. However, for completeness, we have also analyzed the results for power-law relations as indicated above.

3.3.1. Superconducting strips with transport current

A number of situations related to the experimental setup in Fig.7 have been studied, but only a representative set of results are shown below. To start with, the choice of the long strip (tape) geometry allows a straightforward comparison to analytical results for the related limiting critical states. Thus, the thick dashed lines in that figure have been obtained from the well-known expressions [14]

$$J(y) = \begin{cases} \frac{2J_c}{\pi} \tan^{-1} \sqrt{\frac{a^2 - b^2}{b^2 - y^2}} & |y| \leq b \\ J_c & b \leq |y| \leq a, \end{cases} \quad (33)$$

with a the tape “half-thickness” and $b = a\sqrt{(1 - I^2/I_{\max}^2)}$, $I_{\max} \equiv 2aJ_c \equiv$

$2\pi a H_c$. In our case, normalized units have been used for all the physical quantities: $J/J_c, H/H_c, I/I_{\max}, y/a$. The magnetic field profiles have been obtained by integration of the current density. Analytical expressions derived from Eq.(33) may be found in Ref. [14]. As for the results of this paper, straightforward numerical integration of the current density data was performed.

Fig.8 shows the penetration profiles for a transport current that increases from the virgin state until the value $I/I_{\max} = 0.98$ is reached. Then, a negative ramp towards the value $I/I_{\max} = -0.98$ was applied. The associated field penetration profiles are shown in the lower panels. Just for the readers' sake, we recall that expressions for critical state curves in the negative ramp of transport current may be obtained from Eq.(33) by applying linear superposition [14]. For the numerical curves in this plot, the calculated relaxation takes place under the condition $\tau_0/\tau_\rho = 2$ and we just show the profiles previous to the application of the following step (evolution takes place in the fashion shown in Fig.7). Notice that for the increasing branch, a high degree of coincidence between the “relaxed” profiles and the exact critical states displays. It is

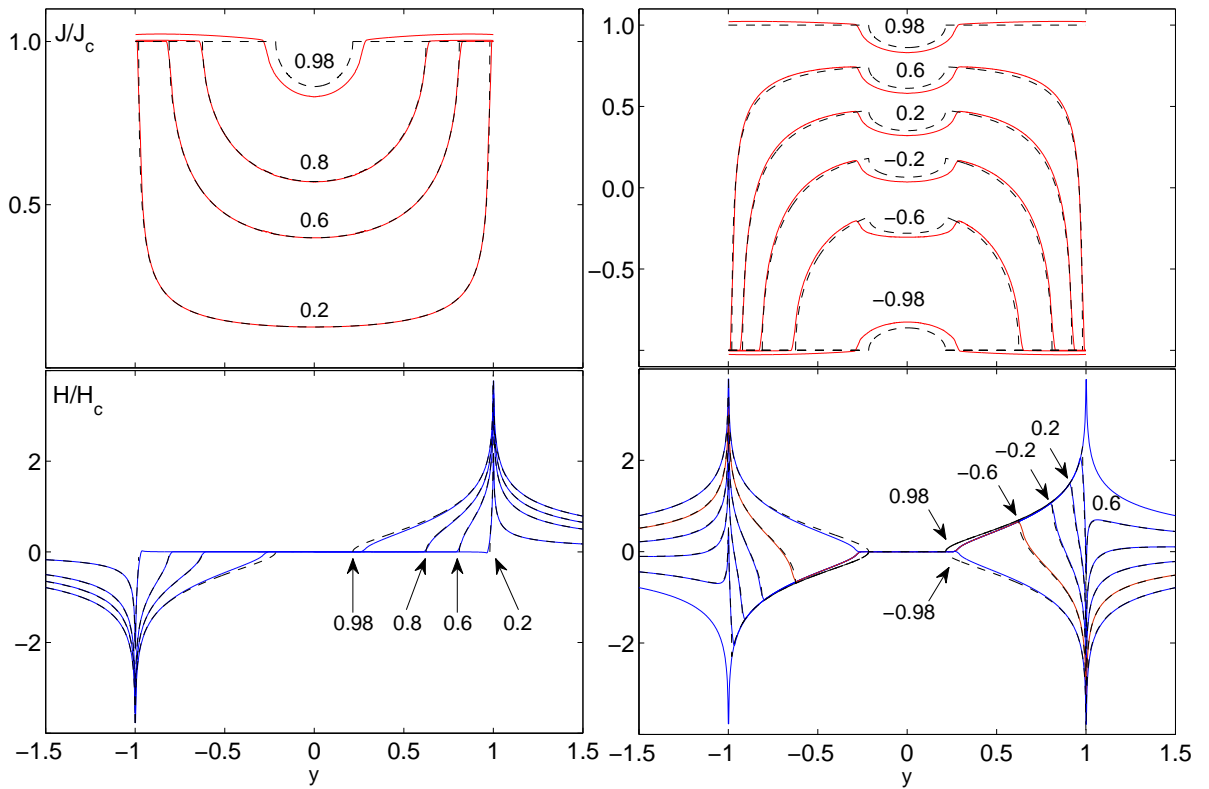


Figure 8. (Color online) Normalized sheet current (top) and magnetic field (bottom) profiles in a superconducting strip carrying a transport current I as labelled in the curves (see text for the definition of units). Dashed lines correspond to the analytical solutions in Ref. [14], and continuous lines to our numerical diffusion calculations. In this case, we plot the profiles obtained for a quadratic dissipation function with $\tau_0/\tau_\rho = 2.0$ just previous to the subsequent step in transport current.

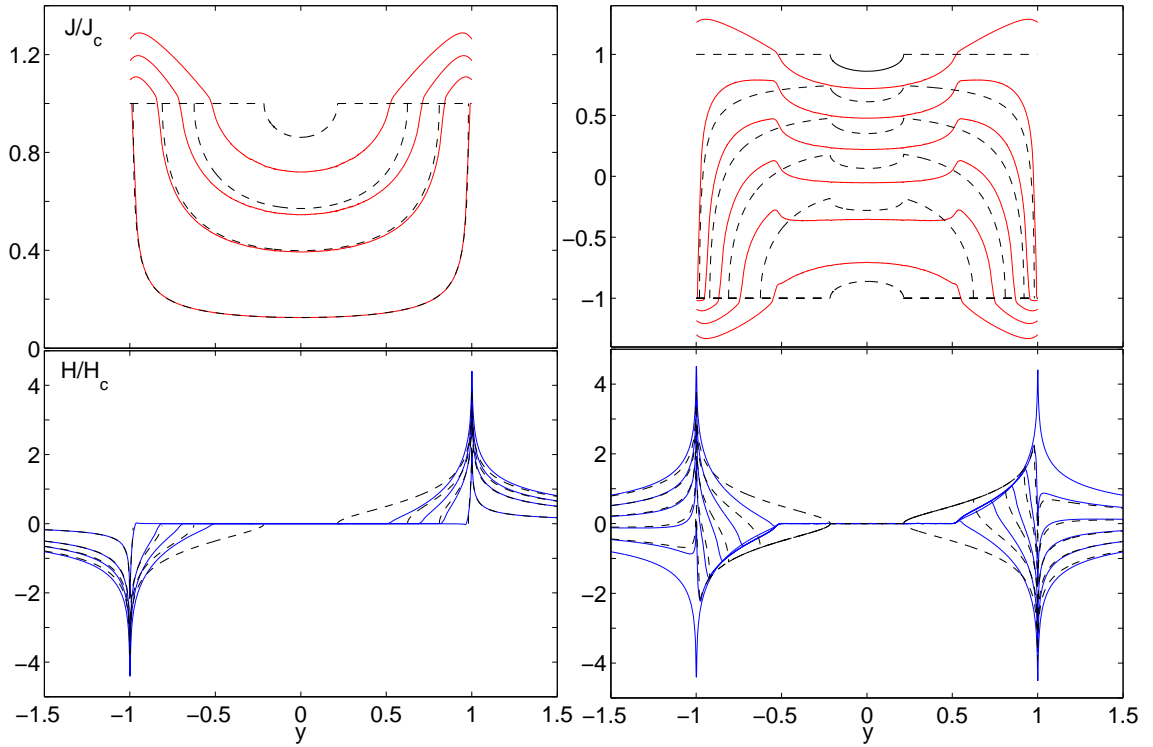


Figure 9. (Color online) Same as Fig.8, but now for $\tau_0/\tau_\rho = 0.2$.

only for the nearly penetrated sample that relaxation is clearly incomplete. A mismatch appears in the central part of the tape, that is inherited by the decreasing branch of the cycle. As a counterpart, a good coincidence is always observed at the sample edges. Apparently, this behavior relates to the concept that flux penetrates from the surface, and that dissipation is an integral over the sample, thus giving place to a most effective relaxation when a smaller region is involved.

Fig.9 has been obtained in a similar fashion, but under the condition $\tau_0/\tau_\rho = 0.2$. As expected, clearly incomplete relaxation is obtained. In practice, this would lead to magnetization profiles with absolute values beyond the critical state limit, and also to higher AC losses, corresponding to the noticeable excursions of the transport current density towards the region $J > J_c$.

We conclude this part by mentioning that results obtained for the power law relation are rather similar to the previously described behavior, when the respective choices $m = 100$ and $m = 10$ are done.

3.3.2. Superconducting strips with applied magnetic field

The diffusing current and field profiles in a thin strip in a perpendicular field H_0 and with zero transport current have been calculated under a wide set of conditions. Figs.10 and 11 present the main features observed in our simulations. On the other

hand, here, the seed for the analytical evaluations is the expression

$$J(y) = \begin{cases} \frac{2J_c}{\pi} \tan^{-1} \frac{cy}{\sqrt{b^2 - y^2}} & |y| \leq b \\ J_c \frac{y}{|y|} & b \leq |y| \leq a, \end{cases} \quad (34)$$

where $c \equiv \tanh(H_0/H_c)$. Magnetic field profiles may be obtained by integration and negative ramp equations by linear superposition [14]. As before, dashed lines stand for the analytical results and continuous curves for our numerical calculations. Again, we only display the profiles corresponding to the diffusion step just previous to the change of external condition.

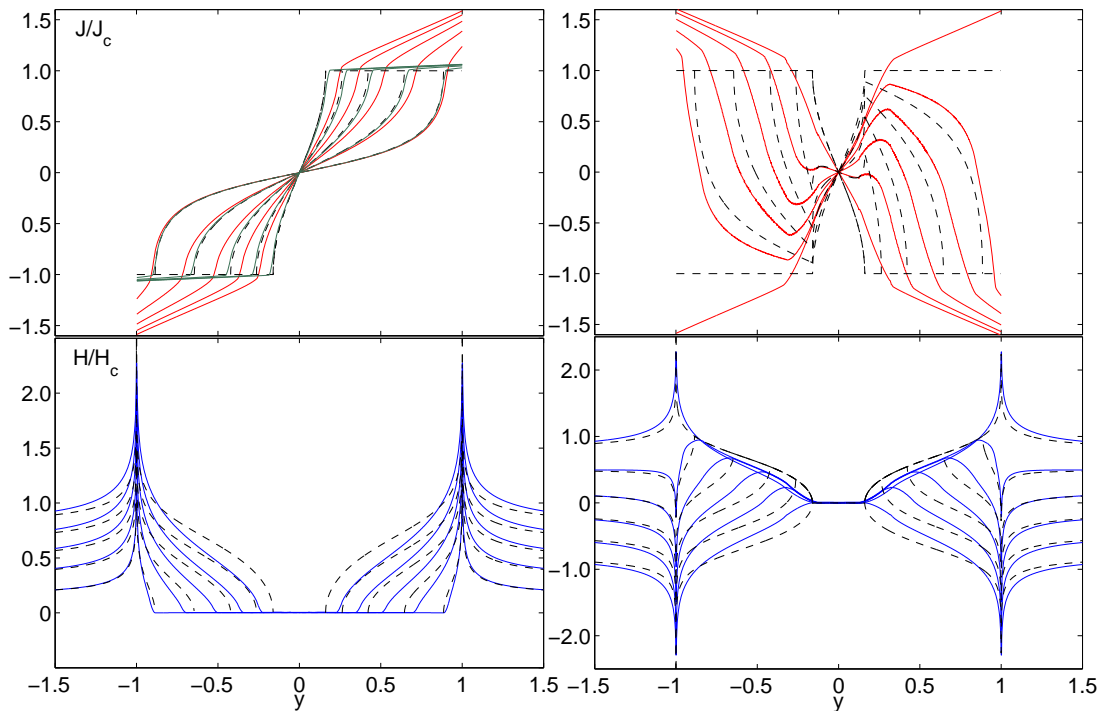


Figure 10. (Color online) Normalized sheet current (top) and magnetic field (bottom) profiles for a superconducting strip in a perpendicular magnetic field. To the left, the depicted profiles correspond to the values $H_0/H_c = 0.16, 0.32, 0.48, 0.64, 0.8$. To the right, the applied field is cycled, i.e.: $H_0/H_c = 0.8, 0.48, 0.16, -0.16, -0.48, -0.8$. Dashed lines correspond to the analytical solutions in Ref. [14], and continuous lines to our numerical diffusion calculations. In this case, we plot the profiles obtained for a quadratic dissipation function with $\tau_0/\tau_\rho = 0.2$ just previous to the subsequent step in transport current. The upper left plot also includes the results for $\tau_0/\tau_\rho = 2.0$, that are omitted in the rest to avoid confusion, and virtually coincide with the analytical lines.

Fig.10 shows the results for the quadratic dissipation function and Fig.11 for the power-law relation. It is noticeable that in both cases, a remarkable degree of coincidence between the relaxed profiles and the eventual critical state solution occurs for $\tau_0/\tau_\rho \geq 2$. Nevertheless, important differences appear for the case $\tau_0/\tau_\rho \simeq 0.2$. As related to such

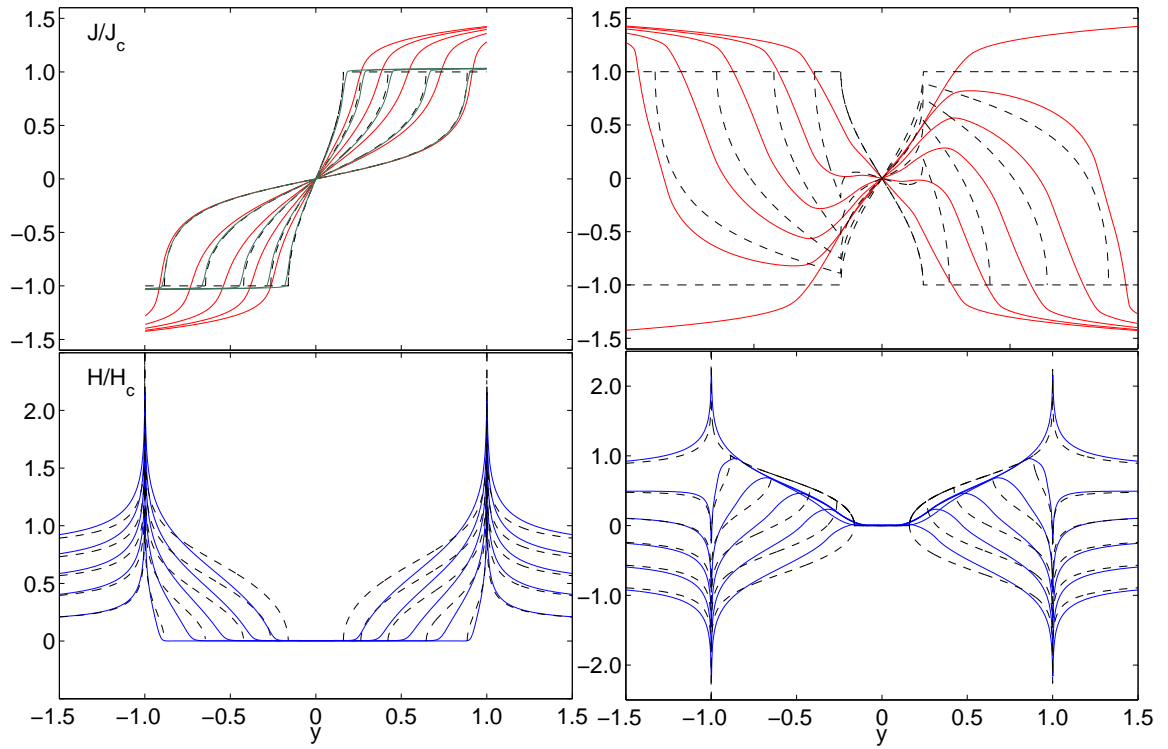


Figure 11. (Color online) Same as Fig.10 but for the power law model with $m = 10$ and $m = 100$.

differences, an important feature has been observed. We call the readers' attention that in Fig.10 $J(y)$ displays a basically linear penetration profile close to the sample's edge, whereas a kind of square root trend is observed in Fig.11. This may be explained as follows. According to Faraday's law, one has $\partial_y E_x = \partial_t B_z$. Then, for small variations $\partial_t B_z$ will be practically constant and so that $E_x \approx (\partial_t H_0) y$, which implies a linear behavior of $J_x(y)$ in the case of a piece-wise linear $E(J)$ law, and an inverse power law when $E \approx J^n$.

3.3.3. Superconducting strips with transport current and applied field

Just for completeness, we will also provide an example of relaxation towards the critical state when both a transport current and a magnetic field are applied to the superconducting strip. Analytical expressions for the case of synchronous ramps of current and field have also been provided in Ref. [14]. Thus, the solution is built through the "generating function"

$$j(y, a, b) = \begin{cases} 1 & b \leq y \leq a \\ \frac{1}{\pi} \cot^{-1} \frac{b^2 - ay}{p} & |y| \leq b \\ 0 & -\infty \leq y \leq -b, \end{cases} \quad (35)$$

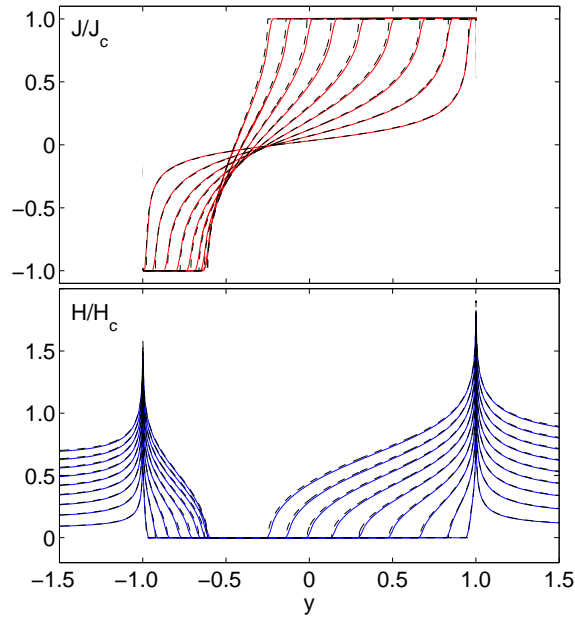


Figure 12. (Color online) Normalized sheet current (top) and magnetic field (bottom) profiles for a “field-like” state in superconducting strip with both transport current and perpendicular magnetic field. In normalized units, the different steps correspond to $H_0/H_c = 0.08, 0.16, \dots, 0.72$ and $I/I_c = 0.025, 0.05, \dots, 0.225$. Dashed lines correspond to the analytical solutions in Ref. [14], and continuous lines to our numerical diffusion calculations. In this case, we plot the profiles obtained for a quadratic dissipation function with $\tau_0/\tau_\rho = 20$.

where $p \equiv \sqrt{(y^2 - b^2)(a^2 - b^2)}$. Explicitly, one has

$$J(y) = J_c [j(y + w, a + w, b) + p j(-y - w, a - w, b)] \quad (36)$$

with the definition $w \equiv (I/I_{\max})/\tanh(H_0/H_c)$.

Fig.12 shows the comparison of our numerical results and the analytical calculation described above. Following the nomenclature in Ref. [14] we have concentrated on a so-called *field-like* situation given by $r \equiv I/2\pi a H_0 = 0.32$.

In this case, we have chosen a high relaxation ratio $\tau_0/\tau_\rho = 20$, and a practical coincidence with the critical state profiles is observed.

4. Conclusions

The motivation of this work has been to explore the possibility of extending the critical state concept so as to include the effects of flux flow resistance in a general sense, i.e.: allowing the possibility of overcritical current density either parallel or

perpendicular to the local magnetic field. Conceptually, we have chosen a macroscopic (thermodynamic) point of view, that avoids the explicit consideration of the underlying vortex physics. In brief, our theory relies on two facts: (i) the experimental evidence that in type-II superconductors dissipativeless currents are allowed when the components of \mathbf{J} either parallel or perpendicular to the local magnetic field do not exceed specific thresholds, and (ii) the resistive transition has to occur according to the laws of entropy production. These considerations are formulated in a geometrical language through the definition of the so-called *dissipation function*, that takes values on the space of current densities, i.e.: $\mathcal{F}(\mathbf{J})$. \mathcal{F} goes to zero within the so-called critical region Δ (defined by the above defined thresholds) and is a positive definite quadratic form beyond. The main role of this function relates to the issue of the material law $\mathbf{E}(\mathbf{J})$ that appears by just imposing consistency and uniqueness in the physical quantities. In fact, one has $\mathbf{E} = \nabla_{\mathbf{J}} \mathcal{F}$.

From the practical point of view, and taking advantage of our variational interpretation of the electromagnetic problem [15], we put forward a minimization statement that gives way to the numerical form of our theory. Eq.(25) displays the function that is minimized, and is the central result of this article. One can identify two basic contributions that relate to the physics of the problem: (i) the inertial terms that account for the reversible energy storage, and (ii) the energy dissipation term that includes \mathcal{F} . Inspired by this interpretation, we have worked out a number of examples that illustrate the application of the theory either straightforwardly to the standard critical state problem, to steady states in which permanent dissipation is forced by some external action, or to the transient (diffusive) processes that occur in between successive critical states.

Based on the comparison of our results and the available literature, we conclude that the complex non-linear diffusion processes that take place according to the theory converge to the well known critical states for a given set of external conditions. Excellent agreement is observed when the system is allowed to relax within the typical excitation period. Related to the analysis of dissipative steady states, we stress the fact that our theory unifies the standard CSM framework and the results of current-voltage techniques, used to derive the critical current parameters. In this sense, we give a number of relations (Eqs.(30) to (32)) that allow to analyze the multicomponent $\mathbf{E}(\mathbf{J})$ measurements designed for characterizing the critical current behavior of High- T_c superconductors [5].

Although we state that a quadratic dissipation function (and thus a piecewise linear $\mathbf{E}(\mathbf{J})$ law) is the more judicious choice for investigating the behavior close to the critical state, the rather extended use of a power-law relation has been checked against our results with good degree of coincidence.

Further work along the lines of this paper entails the application of the theory to obtain the relaxation profiles in higher dimensional problems, such as thick strips or multicomponent magnetic fields.

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